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Space-Time Directional Lyapunov Exponents for Cellular Automata

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Space-time directional Lyapunov exponents are introduced. They describe the maximal velocity of propagation to the right or to the left of fronts of perturbations in a frame moving with a given velocity. The continuity of these exponents as function of the velocity and an inequality relating them to the directional entropy is proved.

KEY WORDS: space-time directional Lyapounov exponents, directional entropy, cellular automata

1. INTRODUCTION

In the theory of smooth dynamical systems Lyapunov exponents describe local instability of orbits. For spatially extended systems, the local instability of patterns is caused by the evolution of localised perturbations. In one dimensional extended systems, the localised perturbations may propagate to the left or to the right not only as travelling waves, but also as various structures. Moreover, other phenomena, called convective instability, have been observed in a fluid flow in a pipe, where it has been found that the system propagates a variety of isolated and localised structures (or patches of turbulence) moving down the pipe along the stream with some velocity. Convective instability has been studied by many authors in various fields (see for example Ref. 2). We introduce Lyapunov exponents describing the maximal velocity of propagation to the right or to the left of fronts of perturbations

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in a frame moving with a given velocity. We consider this problem in the framework of one dimensional cellular automata.

Cellular automata, first introduced by von Neumann have been recently used as mathematical models of natural phenomena.[1,11] Lyapunov exponents in cellular automata were first introduced by Wolfram.[11] The idea was to find a characteristic quantity of the instability of the dynamics of cellular automata analogous to the Lyapunov exponents which measure the instability of the orbits of differentiable dynamical systems under perturbations of initial conditions.

The first rigorous mathematical definition of these exponents was given by Shereshevsky[9] in the framework of ergodic theory.

For a given shift-invariant probability measure μ on the configuration space X that is also invariant under a cellular automaton map f he defined left (resp. right) Lyapunov exponents $\lambda^+(x)$, $\lambda^-(x)$ as maximal time asymptotic speed of propagation to the left (resp. right) of a front of right (resp. left) perturbations of a given configuration $x \in X$. He also gave the following relation between the entropy and the Lyapunov exponents:

$$h_\mu(f) \leq \int_X h_\mu(\sigma, x) \cdot (\lambda^+(x) + \lambda^-(x))\mu(dx)$$

where $h_\mu(\sigma, x)$ denotes the local entropy of the shift σ in x.

In Ref. 10, another slightly different Lyapunov exponents were defined for cellular automata.

The rich structure of cellular automata is best seen when they are considered as dynamical systems associated with continuous maps commuting with the shift. In physical terms this reflects the local nature and the invariance under spatial translations of the interactions between cells, that is a common property of large extended systems in many natural applications.

In order to account of the space-time complexity of cellular automata, Milnor introduced in Ref. 7 a generalization of the dynamical entropy which he called the directional entropy. This concept was later enlarged[3,6] to $\mathbb{Z}^2$-actions on arbitrary Lebesgue spaces.

Here we introduce the notion of space-time directional Lyapunov exponents which are generalizations of the notions considered by Shereshevsky. We define them as the averages along a given space-time direction $\vec{v} \in \mathbb{R} \times \mathbb{R}^+$ of propagation to the left (resp. right) of a front of right (resp. left) perturbations of a given configuration $x \in X$. We compare these exponents with the the directional entropy of the action generated by σ and f (see in Refs. 3, 6 and 7 for the definition and basic properties) and we show, among other things, their continuity. As a corollary to our main result we obtain the estimation of the directional entropy given in Ref. 5.

2. DEFINITIONS AND AUXILIARY RESULTS

Let $X = S^{\mathbb{Z}}$, $S = \{0, 1, \dots, p-1\}$, $p \geq 2$ and let $\mathcal{B}$ be the σ-algebra generated by cylinders. We denote by σ the left shift transformation of X and by f the automaton transformation generated by an automaton rule F, i.e.,

$$(\sigma x)_i = x_{i+1}, \quad (f x)_i = F(x_{i+l}, \dots, x_{i+r}), \quad i \in \mathbb{Z},$$

$$F : S^{r-l+1} \longrightarrow S, \quad l, r \in \mathbb{Z}, \quad l \leq r.$$

Let μ be a probability measure invariant w.r. to σ and f.

Following Shereshevsky[9] we put

$$W_s^+(x) = \{y \in X; y_i = x_i, i \geq s\},$$

$$W_s^-(x) = \{y \in X; y_i = x_i, i \leq -s\},$$

$x \in X, s \in \mathbb{Z}$.

It is easy to see that

(1) The sequences $(W_s^{\pm}(x), s \in \mathbb{Z})$ are increasing, $x \in X$.
(2) For any $a, b, c \in \mathbb{Z}$, $x \in X$ it holds

$$\sigma^a W_c^{\pm}(\sigma^b x) = W_{c \mp a}^{\pm}(\sigma^{a+b} x).$$

Lemma 1. *For any $n \in \mathbb{N}$ we have*

$$f^n\left(W_0^+(x)\right) \subset W_{-nl}^+(f^n x),$$

$$f^n\left(W_0^-(x)\right) \subset W_{nr}^-(f^n x).$$

Proof: It is enough to show the first inclusion. Let $n = 1$ and let $y \in W_0^+(x)$. Hence

$$F(y_{i+l}, \dots, y_{i+r}) = F(x_{i+l}, \dots, x_{i+r})$$

for all $i \geq -l$, i.e.

$$[f(y)]_i = [f(x)]_i, \quad i \geq -l,$$

which means that $f(y) \in W_{-l}^+(f(x))$.

Suppose now that

$$f^n\left(W_0^+(x)\right) \subset W_{-nl}^+(f^n x)$$

for some $n \in \mathbb{N}$. Using (2) one obtains

$$f^{n+1}\left(W_0^+(x)\right) \subset f(W_{-nl}^+(f^n x)) = f(\sigma^{nl} W_0^+(\sigma^{-nl} f^n x))$$
$$\subset \sigma^{nl} W_{-l}^+(f(\sigma^{-nl} f^n x)) = \sigma^{nl} W_{-l}^+(\sigma^{-nl} f^{n+1} x)$$
$$= \sigma^{(n+1)l} W_0^+(\sigma^{-(n+1)l} f^{n+1} x) = W_{-(n+1)l}^+(f^{n+1} x)$$

and so the desired inequality is satisfied for any $n \in \mathbb{N}$. □

Lemma 1 and (1) imply at once the following

Corollary 1. *For any $n \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that*

$$f^n\left(W_0^{\pm}(x)\right) \subset W_s^{\pm}(f^n x).$$

Indeed, applying (1) it is enough to take $s = \max(0, -nl)$ in the case of $W^+(x)$ and $s = \max(0, nr)$ in the case of $W_0^-(x)$.

Let (cf. Ref. 9)

$$\widetilde{\Lambda}_n^{\pm}(x) = \inf\left\{s \geq 0; f^n\left(W_0^{\pm}(x)\right) \subset W_s^{\pm}(f^n x)\right\}$$

and

$$\Lambda_n^{\pm}(x) = \sup_{j \in \mathbb{Z}} \widetilde{\Lambda}_n^{\pm}(\sigma^j x), x \in X, n \in N$$

. Obviously we have

$$(3) \quad 0 \leq \Lambda_n^+(x) \leq \max(0, -nl), \quad 0 \leq \Lambda_n^-(x) \leq \max(0, nr).$$

It is shown in Ref. 9, In the case $l = -r$, $r \geq 0$ that the limits

$$\lambda^{\pm}(x) = \lambda^{\pm}(f; x) = \lim_{n \to \infty} \frac{1}{n} \Lambda_n^{\pm}(x)$$

exist a.e. and they are f (and of course σ)—invariant and integrable.

The proof given in Ref. 9 also works for arbitrary $l, r \in \mathbb{Z}$, $l \leq r$.

The limit λ^+ (resp. λ^-) is called the right (left) Lyapunov exponent of f.

It follows at once from (3) that

$$(4) \quad 0 \leq \lambda^+(x) \leq \max(0, -l), \quad 0 \leq \lambda^-(x) \leq \max(0, r).$$

3. DIRECTIONAL LYAPUNOV EXPONENTS

Let now $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. We put

$$\alpha(t) = [at], \quad \beta(t) = [bt], \quad t \in \mathbb{N}$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$.

It is clear that

$$\alpha(t_1+t_2) \geq \alpha(t_1)+\alpha(t_2), \quad \beta(t_1+t_2) \geq \beta(t_1)+\beta(t_2), \quad t_1, t_2 \in \mathbb{N}$$

and

$$\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = \frac{b}{a}, \quad a \neq 0.$$

We put

$$z_l = a + bl, \quad z_r = a + br.$$

Proposition 1. *For any $t \in \mathbb{N}$ we have*

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^+(x) \subset W^+_{-\alpha(t)-\beta(t)\cdot l}\left(\sigma^{\alpha(t)} f^{\beta(t)} x\right),$$

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^-(x) \subset W^-_{\alpha(t)+\beta(t)\cdot r}\left(\sigma^{\alpha(t)} f^{\beta(t)} x\right).$$

Proof: It follows from Lemma 1 that

$$f^{\beta(t)} W_0^+(x) \subset W^+_{-\beta(t)\cdot l}\left(f^{\beta(t)} x\right).$$

Applying (2) we get

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^+(x) \subset \sigma^{\alpha(t)} W^+_{-\beta(t)\cdot l}\left(f^{\beta(t)} x\right) = W^+_{-\alpha(t)-\beta(t)\cdot l}\left(\sigma^{\alpha(t)} f^{\beta(t)} x\right).$$

Similarly one obtains the second inclusion. □

Corollary 2. *For any $t \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that*

$$\sigma^{\alpha(t)} f^{\beta(t)} W_0^\pm(x) \subset W_s^\pm\left(\sigma^{\alpha(t)} f^{\beta(t)} x\right).$$

We put

$$\widetilde{\Lambda}^\pm_{\vec{v},t}(x) = \inf\left\{s \geq 0; \sigma^{\alpha(t)} f^{\beta(t)} W_0^\pm(x) \subset W_s^\pm\left(\sigma^{\alpha(t)} f^{\beta(t)} x\right)\right\}$$

and

$$\Lambda^\pm_{\vec{v},t}(x) = \sup_{j\in\mathbb{Z}} \widetilde{\Lambda}^\pm_{\vec{v},t}(\sigma^j x).$$

It is clear that

$$(5) \quad 0 \leq \Lambda^+_{\vec{v},t}(x) \leq \max(-\alpha(t) - \beta(t)\cdot l, 0),$$

$$0 \leq \Lambda^-_{\vec{v},t}(x) \leq \max(\alpha(t) + \beta(t)\cdot r, 0).$$

Definition 1. *The function $\lambda^{+}_{\vec{v}}$ (resp. $\lambda^{-}_{\vec{v}}$) defined by the formula*

$$\lambda^{\pm}_{\vec{v}}(x) = \overline{\lim_{t\to\infty}} \frac{1}{t} \Lambda^{\pm}_{\vec{v},t}(x), \quad x \in X$$

is said to be the right (resp. left) space-time directional Lyapunov exponent of f.

We show in the sequel that in fact the limit $\lim_{t\to\infty} \frac{1}{t}\Lambda^{\pm}_{\vec{v},t}(x)$ exists a.e. It follows at once from (5) that

$$(6) \quad 0 \le \lambda^{+}_{\vec{v}}(x) \le \max(-z_l, 0), \quad 0 \le \lambda^{-}_{\vec{v}}(x) \le \max(z_r, 0).$$

Example 1. *We now consider permutative automata. Recall that an automaton map f defined by the rule $F : S^m \to S$ is right permutative if for any $(\bar{x}_1, \dots, \bar{x}_{m-l})$ the mapping: $x_m \mapsto f(\bar{x}_1, \dots, \bar{x}_{r-l}, x_m)$ is one-to-one. A left permutative mapping is defined similarly. The map f is said to be bipermutative if it is right and left permutative.*

Let μ be the uniform Bernoulli measure on X. It is well known that, due to the permutativity of f (right or left), it is f-invariant. It is also σ-invariant. Since the functions $\Lambda^{\pm}_{\vec{v},t}$ are σ-invariant, the ergodicity of μ implies they are constant a.e.

First we consider f being left permutative. It follows straightforwardly from the definitions that

$$\Lambda^{+}_{\vec{v},t} = \max(-\alpha(t) - \beta(t)\cdot l, 0)$$

and so

$$\lambda^{+}_{\vec{v}} = \max(-a - b\cdot l, 0) = \max(-z_l, 0).$$

Now if $\vec{v}$ is such that $z_r = a + br < 0$ we get

$$\lambda^{-}_{\vec{v}} = 0.$$

Indeed, in this case $\alpha(t) + \beta(t)\cdot r < 0$ for sufficiently large t, say $t \ge t_0$ and therefore

$$\widetilde{\Lambda}^{-}_{\vec{v},t} = 0, t \ge t_0.$$

Applying the continuity of the mapping $\vec{v} \mapsto \lambda^{-}_{\vec{v}}$ (Proposition 3) we have $\lambda^{-}_{\vec{v}} = 0$ for $\vec{v}$ such that $z_r = a + br \le 0$. Similarly one checks that for f being right permutative we have

$$\lambda^{-}_{\vec{v}} = \max(z_r, 0)$$

and

$$\lambda_{\vec{v}}^{+} = 0$$

for $\vec{v} = (a, b)$ such that $z_l = a + bl \geq 0$. Thus if f is bipermutative we have :

$$\lambda_{\vec{v}}^{+} = \max(-z_l, 0), \lambda_{\vec{v}}^{-} = \max(z_r, 0)$$

for any $\overrightarrow{v} \in \mathbb{R} \times \mathbb{R}^{+}$.

It easily follows from the definition the following

Lemma 2. *The function $\vec{v} \longrightarrow \Lambda_{\vec{v}}^{\pm}$ is positively homogeneous, i.e. for any $c \in \mathbb{R}^{+}$ we have*

$$\lambda_{c\vec{v}}^{\pm} = c\lambda_{\vec{v}}^{\pm}.$$

Lemma 3. *For any $\vec{v} \in \mathbb{R} \times \mathbb{R}^{+}$, $s, t \in \mathbb{N}$ and $x \in X$ we have*

$$\Lambda_{\vec{v},s+t}^{\pm}(x) \leq \Lambda_{\vec{v},s}^{\pm}(x) + \Lambda_{\vec{v},t}^{\pm}\left(f^{\beta(s)}x\right) + 2|l|.$$

Proof: We shall consider only the case of $\Lambda_{\vec{v},s+t}^{+}$; the proof for $\Lambda_{\vec{v},s+t}^{-}$ is similar, $s, t \geq 0$.

We put

$$\widetilde{s} = \Lambda_{\vec{v},s}^{+}(x), \quad \widetilde{t} = \Lambda_{\vec{v},t}^{+}\left(f^{\beta(s)}x\right).$$

By the definition and the σ-invariance of $\Lambda_{\vec{v},t}^{+}$ we have

$$\begin{aligned}(7) \quad \sigma^{\alpha(s)+\alpha(t)} f^{\beta(s)+\beta(t)} W_0^{+}(x) &\subset \sigma^{\alpha(t)} f^{\beta(t)} W_{\widetilde{s}}^{+}\left(\sigma^{\alpha(s)} f^{\beta(s)} x\right) \\ &= \sigma^{\alpha(t)} f^{\beta(t)} \sigma^{-\widetilde{s}} W_0^{+}\left(\sigma^{\widetilde{s}+\alpha(s)} f^{\beta(s)} x\right) \\ &\subset \sigma^{-\widetilde{s}} W_{\widetilde{t}}^{+}\left(\sigma^{\widetilde{s}} \sigma^{\alpha(s)+\alpha(t)} f^{\beta(s)+\beta(t)} x\right) \\ &= W_{\widetilde{s}+\widetilde{t}}^{+}\left(\sigma^{\alpha(s)+\alpha(t)} f^{\beta(s)+\beta(t)} x\right).\end{aligned}$$

We put

$$\delta_\alpha = \alpha(s+t) - (\alpha(s) + \alpha(t)), \quad \delta_\beta = \beta(s+t) - (\beta(s) + \beta(t)), s, t \geq 0.$$

Letting $\sigma^{\delta_\alpha} f^{\delta_\beta}$ act on both sides of (7) we obtain by Lemma 1

$$\begin{aligned}(8) \quad \sigma^{\alpha(s+t)} f^{\beta(s+t)} W_0^{+}(x) &\subset \sigma^{\delta_\alpha} f^{\delta_\beta} W_{\widetilde{s}+\widetilde{t}}^{+}\left(\sigma^{\alpha(s)+\alpha(t)} f^{\beta(s)+\beta(t)} x\right) \\ &\subset W_{\widetilde{s}+\widetilde{t}-\delta_\alpha-\delta_{\beta}\cdot l}^{+}\left(\sigma^{\alpha(s+t)} f^{\beta(s+t)} x\right).\end{aligned}$$

Since $0 \leq \delta_\alpha, \delta_\beta \leq 2$ we get from (8)

$$\begin{aligned}\widetilde{\Lambda}^{+}_{\vec{v},s+t}(x) &\leq \max\left(\widetilde{s} + \widetilde{t} - \delta_\alpha - \delta_\beta l, 0\right)\\ &\leq \widetilde{s} + \widetilde{t} + 2|l|\\ &= \Lambda^{+}_{\vec{v},s}(x) + \Lambda^{+}_{\vec{v},t}\left(f^{\beta(s)}x\right) + 2|l|\end{aligned}$$

which implies at once the desired inequality. □

Proposition 2. *For any $\vec{v} \in \mathbb{R} \times \mathbb{R}^+$ and almost all $x \in X$ there exists the limit*

$$\lim_{t\to\infty} \frac{1}{t} \Lambda^{\pm}_{\vec{v},t}(x) = \lambda^{\pm}_{\vec{v}}(x).$$

Proof: It is enough to consider the case of $\Lambda^{+}_{\vec{v},t}$, $t \geq 0$.

First we consider the case $\vec{v} = (a, 1)$,

i.e.,

$$\alpha(t) = [at], \quad \beta(t) = t, \quad t > 0.$$

In this case Lemma 3 has the form

$$\Lambda^{\pm}_{\vec{v},s+t}(x) \leq \Lambda^{\pm}_{\vec{v},s}(x) + \Lambda^{\pm}_{\vec{v},t}(f^s x) + 2|l|, s, t \geq 0.$$

It is easy to see that this inequality permits to apply the Kingman subadditive ergodic theorem, which implies the existence a.e. of the limit

$$\lim_{t\to\infty} \frac{1}{t} \Lambda^{\pm}_{\vec{v},t}(x) = \lambda^{\pm}_{\vec{v}}(x).$$

The case of arbitrary $\vec{v}$ easily reduces to the above by Lemma 2. □

Proposition 3. *The space-time directional Lyapunov exponents $\lambda^{\pm}_{\vec{v}}$, are continuous as functions of $\vec{v}$ for $\vec{v} \in \mathbb{R} \times \left(\mathbb{R}^+ \setminus \{0\}\right)$, i.e. $\lim_{n\to\infty} \lambda^{+}_{\vec{v}_n}(x) = \lambda^{+}_{\vec{v}}(x)$ for any $\vec{v}_n \longrightarrow \vec{v}$ as $n \longrightarrow \infty$ and any $x \in X$.*

Proof: The result will easily follow from the inequality

$$\text{(9)} \quad b \cdot \lambda^{+}_{\vec{v}'}(x) \leq \max\left(b' \lambda^{+}_{\vec{v}}(x) - (a'b - ab'), 0\right)$$

where $\vec{v} = (a, b)$, $\vec{v}' = (a', b')$, $b, b' > 0$.

In order to show (9) it is enough to consider the case $b' = b = 1$. In this case $\beta(t) = \beta'(t) = t$, $t > 0$. We put $p(t) = \alpha'(t) - \alpha(t)$, $t > 0$.

We have

$$\begin{aligned}\sigma^{\alpha'(t)} f^{\beta'(t)} W_0^+(x) &= \sigma^{\alpha(t)+p(t)} f^{\beta(t)} W_0^+(x)\\ &= \sigma^{p(t)} \sigma^{\alpha(t)} f^{\beta(t)} W_0^+(x)\end{aligned}$$

$$\subset \sigma^{p(t)} W^{+}_{\tilde{\Lambda}^{+}_{\vec{v},t}(x)} \left(\sigma^{\alpha(t)} f^{\beta(t)} x\right)$$

$$= W^{+}_{\tilde{\Lambda}^{+}_{\vec{v},t}(x)-p(t)} \left(\sigma^{p(t)} \sigma^{\alpha(t)} f^{\beta(t)} x\right)$$

$$= W^{+}_{\tilde{\Lambda}^{+}_{\vec{v},t}(x)-p(t)} \left(\sigma^{\alpha'(t)} f^{\beta'(t)} x\right)$$

$$\subset W^{+}_{\max(\Lambda^{+}_{\vec{v},t}(x)-p(t),0)} \left(\sigma^{\alpha'(t)} f^{\beta'(t)} x\right), \ t > 0$$

which implies

$$(10) \quad \lambda^{+}_{\vec{v}'}(x) \leq \max\left(\lambda^{+}_{\vec{v}}(x) - (a' - a), 0\right),$$

i.e., (9) is satisfied for $b = b' = 1$.

Let now $\vec{v} = (a, b), \ b > 0$ be fixed and let $\vec{v}'_n = (a'_n, b'_n)$ be such that $\vec{v}'_n \longrightarrow \vec{v}$ as $n \longrightarrow \infty$.

It follows from (9) and (10), respectively, that

$$\overline{\lim_{n\to\infty}} \lambda^{+}_{\vec{v}'_n}(x) \leq \max\left(\lambda^{+}_{\vec{v}}(x), 0\right) = \lambda^{+}_{\vec{v}}(x),$$

$$\lambda^{+}_{\vec{v}}(x) \leq \max\left(\underline{\lim}_{n\to\infty} \lambda^{+}_{\vec{v}'_n}(x), 0\right) = \underline{\lim}_{n\to\infty} \lambda^{+}_{\vec{v}'_n}(x)$$

i.e.,

$$\lim_{n\to\infty} \lambda^{+}_{\vec{v}'_n}(x) = \lambda^{+}_{\vec{v}}(x)$$

which gives the desired result.

Let now Φ be the action generated by σ and f and let $h^{\mu}_{\vec{v}}(\Phi)$ denote the directional entropy of Φ in the direction $\vec{v}$. □

Theorem 1. *For any $\vec{v} = (a, b)$, $a \in \mathbb{R}, \ b \geq 0$ and any Φ-invariant measure μ we have*

$$h^{\mu}_{\vec{v}}(\Phi) \leq \int_X h_{\mu}(\sigma, x)\left(\lambda^{+}_{\vec{v}}(x) + \lambda^{-}_{\vec{v}}(x)\right)\mu(dx)$$

where $h_{\mu}(\sigma, x)$ is the local entropy of σ at the point x.[9] *In particular, if μ is ergodic with repect to σ, then $\lambda^{\pm}_{\vec{v}}$ are constant a.e. and*

$$h^{\mu}_{\vec{v}}(\Phi) \leq h_{\mu}(\sigma)\left(\lambda^{+}_{\vec{v}} + \lambda^{-}_{\vec{v}}\right).$$

Proof: First we consider the case $\vec{v} = (p, q) \in \mathbb{Z} \times \mathbb{N}$. In this case it is easy to show that

$$h^{\mu}_{\vec{v}}(\Phi) = h_{\mu}(\sigma^p f^q), \quad \lambda^{+}_{\vec{v}}(x) = \lambda^{+}(\sigma^p f^q; x).$$

Since $\sigma^p f^q$ is an automaton map, Theorem of Ref. 9 implies

$$
\begin{aligned}
h^{\mu}_{\vec{v}}(\Phi) &= h_\mu(\sigma^p f^q) \\
&\leq \int_X h_\mu(\sigma, x)\left(\lambda^+(\sigma^p f^q; x) + \lambda^-(\sigma^p f^q; x)\right)\mu(dx) \\
&= \int_X h_\mu(\sigma, x)\left(\lambda^+_{\vec{v}}(x) + \lambda^-_{\vec{v}}(x)\right)\mu(dx).
\end{aligned} \tag{11}
$$

The homogeneity of the mappings

$$\vec{v} \longrightarrow h^{\mu}_{\vec{v}}(\Phi), \quad \vec{v} \longrightarrow \lambda^+_{\vec{v}}$$

and (11) imply that the inequality

$$h^{\mu}_{\vec{v}}(\Phi) \leq \int_X h_\mu(\sigma, x)\left(\lambda^+_{\vec{v}}(x) + \lambda^-_{\vec{v}}(x)\right)\mu(dx) \tag{12}$$

is valid for every $\vec{v} \in \mathbb{Q} \times \mathbb{Q}^+$.

Let now $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. If $b = 0$ then the desired inequality is satisfied because for $\vec{v} = (1, 0)$ we have $h_{\vec{v}}(\Phi) = h_\mu(\sigma)$, $\lambda^+_{\vec{v}}(x) = 0,\ \lambda^-_{\vec{v}}(x) = 1,\ x \in X$. Thus let us suppose $b > 0$ and let $\vec{v}_n = (a_n, b_n) \longrightarrow \vec{v}$. Hence and from the inequalities

$$0 \leq \lambda^+_{\vec{v}_n}(x) \leq \max\left(0, -z_l^{(n)}\right), \qquad 0 \leq \lambda^-_{\vec{v}_n}(x) \leq \max\left(0, -z_r^{(n)}\right)$$

where $z_l^{(n)} = a_n + b_n l,\ z_r^{(n)} = a_n + b_n r,\ n \geq 1$ it follows that the sequence $(\lambda^+_{\vec{v}_n}(x) + \lambda^-_{\vec{v}_n}(x))$ is jointly bounded. By Proposition 3

$$\lambda^{\pm}_{\vec{v}_n}(x) \longrightarrow \lambda^{\pm}_{\vec{v}}(x) \quad \text{a.e.}$$

It follows from Ref. 8 that the mapping $\vec{v} \longrightarrow h^{\mu}_{\vec{v}}(\Phi)$ is continuous. It is well known (cf. Ref. 4) that the function $x \longrightarrow h_\mu(\sigma, x)$ is integrable.

Therefore applying the Lebesgue dominated convergence theorem we get from (12) the desired inequality for all $\vec{v} \in \mathbb{R} \times \mathbb{R}^+$. The inequality in the ergodic case follows at once from the Brin-Katok formula[4]:

$$\int_X h_\mu(\sigma, x)\mu(dx) = h_\mu(\sigma)$$

The above theorem immediately implies □

Corollary 3. *For any $\vec{v} \in \mathbb{R} \times \mathbb{R}^+$ and for any Φ-invariant measure μ we have*

$$h^{\mu}_{\vec{v}}(\Phi) \leq h_\mu(\sigma)\left(\max(0, -z_l) + \max(0, z_r)\right).$$

It is interesting to see that our computation of the Lyapounov exponents in the example of Section 3 and the results of Ref. 5 (Theorems 1 and 2) imply the following.

Remark 1. *The inequality given in the above theorem (and Corollary 1) becomes an equality in the following cases:*

(i) f *left permutative and* $z_r \leq 0$,
(ii) f *right permutative and* $z_l \geq 0$,
(iii) f *bipermutative and* $z_l \leq 0$, $z_r \geq 0$.

From Corollary 1 one easily obtains the following estimation for $h^{\mu}_{\vec{v}}(\Phi)$ proved in Ref. 5.

Corollary 4. *For any* $\vec{v} \in \mathbb{R} \times \mathbb{R}^+$ *and for any* Φ*-invariant measure* μ *we have*

$$h^{\mu}_{\vec{v}}(\Phi) \leq \max(|z_l|, |z_r|) \log p \quad \text{if} \quad z_l \cdot z_r \geq 0$$

and

$$h^{\mu}_{\vec{v}}(\Phi) \leq |z_r - z_l| \log p \quad \text{if} \quad z_l \cdot z_r \leq 0.$$

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REFERENCES

1. J.-P. Allouche, M. Courbage, J. P. S. Kung and G. Skordev, *Cellular automata, Encyclopedia of Physical Science and Technology* (3rd en. Vol. 2, pp. 555–567, Academic Press).
2. T. Bohr and D. A. Rand, A mechanism for localized turbulence, *Physica D* **52**:532–543 (1991).
3. M. Boyle and D. Lind. Expansive subdynamics. *Trans. Am. Math. Soc.* **349**:55–102 (1997).
4. M. Brin and A. Katok, *On local entropy*. *Lecture Notes in Math.* 1007. (Springer Verlag, Berlin, 1983, pp. 30–38).
5. M. Courbage and B. Kamiński, On the directional entropy of $\mathbb{Z}^2$-actions generated by cellular automata. *Studia Math.* **153**:285–295 (2002).
6. B. Kamiński and K. K. Park, On the directional entropy of $\mathbb{Z}^2$-action on a Lebesgue space. *Studia Math.* **133**:39–51 (1999).
7. J. Milnor, On the entropy geometry of cellular automata. *Complex Syst.* **2**:357–386 (1988).
8. K. K. Park, On directional entropy functions. *Israel J. Math.* **113**:243–267 (1999).
9. M. A. Shereshevsky, Lyapunov exponents for one-dimensional automata. *J. Nonlinear Sci.* **2**:1–8 (1992).
10. P. Tisseur, Cellular automata and Lyapunov exponents. *Nonlinearity* **13**:1547–1560 (2000).
11. S. Wolfram, Cellular automata and complexity. (Addison-Wesley Publishing Company, 1994).